Fibonacci Numbers

Vishal Gupta

Introduction

I first heard about Fibonacci numbers some day in primary school where we got a worksheet with a tower on it, with each floor of the tower having a box to write the next Fibonacci number - it was a challenge to 'scale the tower' by working out as many Fibonacci numbers correctly as possible as some sort of exercise in addition. It was pretty neat that the numbers grew so big (which can be quantified in a neat way as shown later!) but other than that they seemed kind of uninteresting with nothing that remarkable about them. A decent amount of time later I was in year 12 and was looking for stuff to do for my personal statement, having recently discovered online courses. I found this course on Fibonacci numbers by Professor Chasnov which was great, this document is inspired by that course (which is the reason why I care about Fibonacci numbers now!) so I recommend you check that out if you find this sort of stuff cool.

History and Definition

Fibonacci numbers are named after Fibonacci, as expected, and they were first described in Indian Mathematics. In Indian Mathematics, they appear as the number of poems of given length you can form in the Sanskrit poetic tradition of combining long syllables of length two with short syllables of length one. The Fibonacci numbers are defined by

$$F_n = F_{n-1} + F_{n-2} \tag{1}$$

This is known as a recurrence relation. It was first used by Fibonacci in the book Liber Abaci to describe the growth of the population of rabbits in an idealised situation - we start with a pair of newly born rabbits where each pair of rabbits mates at the age of one month, and they produce another pair of rabbits each month. The question then is, what is the number of pairs of rabbits after a year? This can be visualised in the following way:

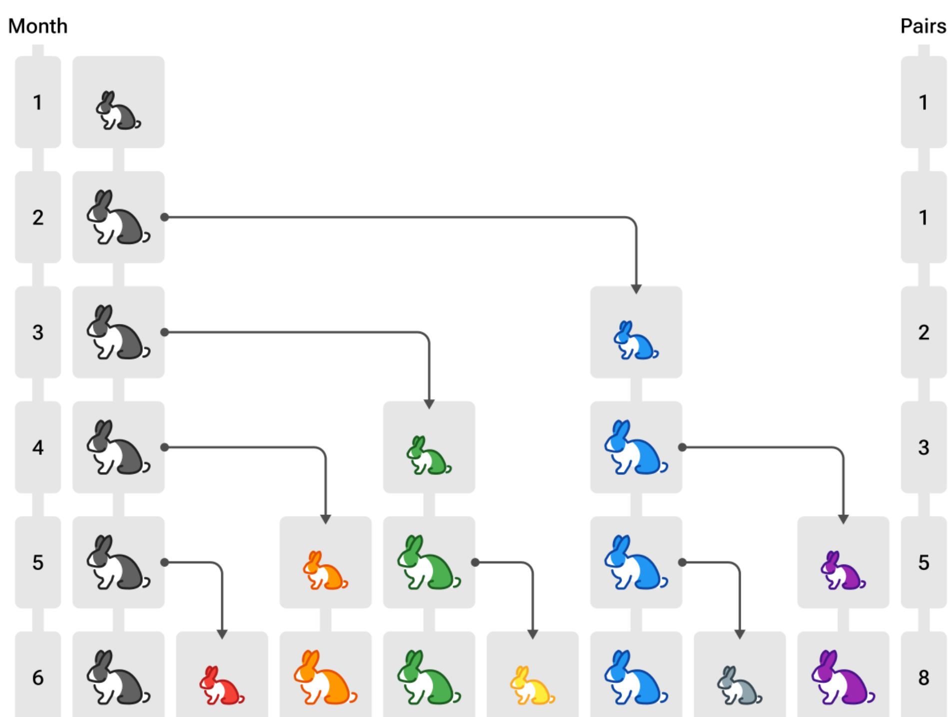


Figure 1: A diagram of the problem (from Wikipedia)

You might like to try solve it yourself, the diagram and its arrows in particular can help you discover the solution. If not, you can think about them in terms of the mature and juvenile of each generation. If we call the number of rabbits in a generation F_n, the number of juveniles J_n and the number of mature rabbits M_n then we

have $F_n = J_n + M_n$. Also, note that in the n^{th} generation, the number of rabbits is the number of rabbits from the preceding generation plus the number of mature rabbits in the previous generation as these are the new rabbits they give birth to so that $F_n = F_{n-1} + M_{n-1}$. The number of mature rabbits in a generation is the number of rabbits in total in the previous generation, the number of mature rabbits from the previous generation which carry over and the juveniles which mature; i.e $M_n = M_{n-1} + J_{n-1} = F_{n-1}$.
Therefore we have $F_n = F_{n-1} + F_{n-2}$ as we hoped for - the Fibonacci sequence.

Some questions you can try:

1. Prove that the Sanskrit poetry problem also results in the Fibonacci sequence. (Hint: Think about placing the first syllable and the resulting cases)
2. How does the previous problem relate to this one 'How many ways are there of covering a 2xn rectangle with 2x1 dominoes'? (Oxford Maths interview question)
3. How many binary sequences are there with no consecutive zeroes?

Relation to the golden ratio φ

The golden ratio $\varphi = \dfrac{1+\sqrt{5}}{2}$ appears a few times throughout mathematics, enough to have its own name. It appears in connection with the Fibonacci numbers, in particular, the growth of Fibonacci numbers can be quantified with

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \varphi$$

To prove this, note that since $F_n = F_{n-1} + F_{n-2}$, we have $F_{n-1} < F_n < 2F_{n-1}$ so that the ratios are all between 1 and 2. Call the limit L. Then since

$$\frac{F_n}{F_{n-1}} = \frac{F_{n-1} + F_{n-2}}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}}$$

by taking $n \to \infty$ in this, we obtain $L = 1 + \dfrac{1}{L}$. This is a quadratic equation upon rearranging; $L^2 - L - 1 = 0$ which has two solutions,

$$L = \frac{1 \pm \sqrt{5}}{2}$$

Since the ratios are all positive, this must be $L = \dfrac{1+\sqrt{5}}{2} = \varphi$, the golden ratio. Note that the other solution is $L = -\varphi^{-1}$.
With this the growth of Fibonacci numbers is approximately $F_n \approx C\varphi^n$ for some constant C independent of n. This isn't the full story of course, what's the constant C? It turns out you can go further than this and say

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

so that $C \approx \dfrac{1}{\sqrt{5}}$ - the φ^{-n} term dies for large n. This is known as Binet's formula and to me is just crazy - the Fibonacci numbers are all integers yet the right hand side is full of square roots of 5 which happen to all magically cancel out but also manage to follow a relation as neat as (1)! To prove this, will require some results from A level Further Maths (methods to solve homogeneous linear second order difference equations), we try a solution $F_n = \lambda^n$ - the justification of this is in IA Differential Equations. Substituting this in, we obtain

$$\lambda^{n+1} = \lambda^n + \lambda^{n-1}$$

which, upon dividing through by λ^{n-1}, gives us the equation $\lambda^2 - \lambda - 1 = 0$ which is familiar from the previous part with the limit. This then gives the full solution to the difference equation as

$$F_n = C\varphi^n + D(-\varphi)^{-n}$$

The constants C and D are determined by the initial two values in our Fibonacci sequence which in our case gives us Binet's formula - for other initial values you get some other constants. This representation of the Fibonacci numbers highlights an important thing in mathematics, some representations of things make it easier to understand a certain property of the object you're looking at. From the original definition, it's quite clear to see what the relation between each Fibonacci number is and that they're all integers while Binet's formula lets you see the growth of the Fibonacci numbers quite easily and lets you jump straight to something like F_100 without having to calculate the preceding ones.

Continued fractions and flowers

φ can be approximated by ratios of Fibonacci numbers as we saw in the previous section. They turn out to be the best possible rational approximations to φ. To demonstrate this, let's start with the equation we got from calculating the limit from the last section, $\varphi = 1 + \frac{1}{\varphi}$. Repeatedly applying the formula, we obtain

$$\varphi = 1 + \frac{1}{1+\frac{1}{\varphi}} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{\varphi}}} = \dots$$

Doing this forever, φ can be written as the continued fraction

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}$$

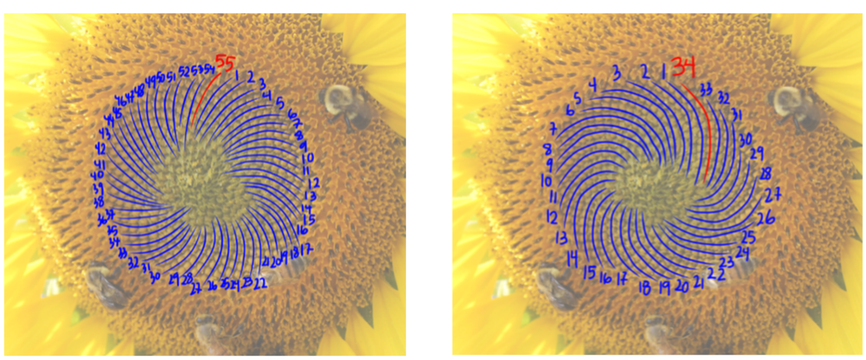
A useful notion when studying continued fractions are convergents - these are what you get when terminating the continued fraction expression at a certain point and checking the rational number you get. In the case of φ, this gives

$$1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \dots$$

You might notice the pattern of these numbers, they're ratios of consecutive Fibonacci numbers! To prove this, it can be done by induction, the n^{th} convergent C_n can be written as

$$C_n = 1 + \frac{1}{\frac{F_{n-1}}{F_{n-2}}} = 1 + \frac{F_{n-2}}{F_{n-1}} = \frac{F_n}{F_{n-1}}$$

This gives another proof of $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \varphi$ notably.
This is important because a theorem from Number Theory (proven in II Number Theory) is the law of best approximations, for a given irrational r, the best rational approximation to r with denominator less than some number q is $\frac{a}{b}$ where this fraction is a convergent of the continued fraction.
For example, with φ, the best approximation with denominator less than or equal to 5 is $\frac{8}{5}$ while for 8 it's $\frac{13}{8}$ etc.
Since φ's continued fraction is full of 1s, this means that it's the irrational which is most difficult to approximate by rationals - its the 'most irrational of irrational numbers'
This actually explains the spirals in the heads of flowers such as sunflowers which is a really neat result:



Modified: Image Credit: Flickr @ Sue Reynolds

Modified: Image Credit: Flickr @ Sue Reynolds

Figure 2: The clockwise and anticlockwise spirals

Notice the clockwise and anticlockwise have 55 and 34 spirals - these are both Fibonacci numbers. The reason for this appearance of the Fibonacci numbers in nature is about the sunflower trying to pack as many seeds as possible in its head. Each seed is produced by starting from the centre and moving some angle away from the previous seed. If this angle is a rational angle $\frac{p}{q}$ then the resulting seed packing will be q straight lines which won't be as efficient as possible. So you want an irrational packing which will cause a spiral - it won't ever overlap itself because it's irrational. However, if this irrational is very closely approximated by a rational $\frac{p}{q}$ then the resulting spirals will be close to the q straight lines. So the best packing is the irrational which is worst approximated by irrationals - namely φ.

Matrices

This section is only relevant if you've studied matrices, if not, you can skip reading this! It's intended more as a challenge though to have a go at if you're interested. The Fibonacci Q matrix can be defined by

$$Q = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Some questions relating to this are:

1. By considering the determinant, prove *Cassini's identity*, that $F_n F_{n-1} - F_n^2 = (-1)^n$ for all integers n.
2. By considering powers of Q and using $Q^n = Q^{n-1}Q$, prove the Fibonacci addition formula,

$$F_m F_n + F_{m-1} F_{n-1} = F_{n+m-1}$$

There is a STEP 2 Question on these (96-S2-Q3) using induction to prove them if you'd like to try that.